Home Work 1
ME/ECE 236 – Spring 2008

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Problem 1 (Text problem 1.16) Given:

$$m\frac{d^2}{dt^2}(x_c + L\sin\theta) = F_x \tag{1}$$

$$m\frac{d^2}{dt^2}(L\cos\theta) = F_y \tag{2}$$

$$I\ddot{\theta} = u + F_y L\sin\theta - F_x L\cos\theta \tag{3}$$

$$M\ddot{x}_c = -F_x - kx_c \tag{4}$$

(a) Carrying out the indicated differentiation and eliminating F_x and F_y, show that the equations of motion reduce to

$$D(\theta)\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = \begin{bmatrix}u\\ mL\dot{\theta}^2\sin\theta - kx_c\end{bmatrix}, D(\theta) = \begin{bmatrix}I + mL^2 & mL\cos\theta\\ mL\cos\theta & M+m\end{bmatrix}$$

Differentiating (1) and (2) yields:

$$F_x = m\frac{d}{dt}(\dot{x}_c + L\dot{\theta}\cos\theta) = m\ddot{x}_c + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta$$

$$F_y = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta$$

Plugging these values for F_x and F_y into (3) yields:

$$I\ddot{\theta} = u + (-mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta)L\sin\theta - (m\ddot{x}_c + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta)L\cos\theta$$

$$\Rightarrow I\ddot{\theta} = u - mL\ddot{x}_c\cos\theta - mL^2\ddot{\theta}(sin^2\theta + cos^2\theta) - mL^2\dot{\theta}^2(\cos\theta\sin\theta - \sin\theta\cos\theta)$$

$$\Rightarrow I\ddot{\theta} = u - mL\ddot{x}_c\cos\theta - mL^2\ddot{\theta}$$

$$\Rightarrow I\ddot{\theta} + mL^2\ddot{\theta} + mL\ddot{x}_c\cos\theta = u$$

$$\begin{bmatrix}I + mL^2 & mL\cos\theta\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = [u] \tag{5}$$

Plugging above value for F_x into (4) yields:

$$M\ddot{x}_c = -(m\ddot{x}_c + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta) - kx_c$$

$$\Rightarrow mL\ddot{\theta}\cos\theta + M\ddot{x}_c + m\ddot{x}_c = mL\dot{\theta}^2\sin\theta - kx_c$$

$$\begin{bmatrix} mL\cos\theta & M+m \end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = [mL\dot{\theta}^2\sin\theta - kx_c] \tag{6}$$

Combining (5) and (6) yields:

$$\begin{bmatrix} I+mL^2 & mL\cos\theta \\ mL\cos\theta & M+m \end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = \begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c\end{bmatrix}$$

$$\Rightarrow D(\theta)\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = \begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c\end{bmatrix}, D(\theta) = \begin{bmatrix} I+mL^2 & mL\cos\theta \\ mL\cos\theta & M+m \end{bmatrix}$$

(b) Solving the foregoing equation for $\ddot{\theta}$ and $\ddot{x}_c$, show that

$$\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = \frac{1}{\Delta(\theta)}\begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}\begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c\end{bmatrix}$$

where

$$\Delta(\theta) = (I+mL^2)(m+M) - m^2L^2\cos^2\theta \geq (I+mL^2)M + mI > 0$$

$$D(\theta)^{-1} = \frac{1}{(\det D(\theta))}\begin{bmatrix} M+m & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}$$

$$\Rightarrow D(\theta)^{-1} = \frac{1}{((I+mL^2)(m+M) - m^2L^2\cos^2\theta)}\begin{bmatrix} M+m & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}$$

$$\Rightarrow D(\theta)^{-1} = \frac{1}{\Delta(\theta)}\begin{bmatrix} M+m & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}$$

Left-multiplying the conclusion of part (a) on both sides by $\Rightarrow D(\theta)^{-1}$ on both sides yields:

$$D(\theta)^{-1}D(\theta)\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = D(\theta)^{-1}\begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c\end{bmatrix}$$

$$\begin{bmatrix}\ddot{\theta}\\ \ddot{x}_c\end{bmatrix} = \frac{1}{\Delta(\theta)}\begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}\begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c\end{bmatrix} \tag{7}$$

(c) Using $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x_c$, and $x_4 = \dot{x}_c$ as the state variables and u as the control input, write down the state equation

$$\dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = \ddot{\theta} = \frac{1}{\Delta(\theta)}(um + uM - m^2L^2\dot{\theta}^2 \cos\theta \sin\theta + mLkx_c \cos\theta)$$

$$\Rightarrow \dot{x}_2 = \frac{1}{\Delta(x_1)}(um + uM - m^2L^2x_2^2 \cos x_1 \sin x_1 + mLkx_3 \cos x_1)$$

$$\dot{x}_3 = \dot{x}_c = x_4$$

$$\dot{x}_4 = \ddot{x}_c = \frac{1}{\Delta(x_1)}(-mLu \cos x_1 + (I + mL^2)(mLx_2^2 \sin x_1 - kx_3)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{\Delta(x_1)}(um + uM - m^2L^2x_2^2 \cos x_1 \sin x_1 + mLkx_3 \cos x_1 \\ x_4 \\ \frac{1}{\Delta(x_1)}(-mLu \cos x_1 + (I + mL^2)(mLx_2^2 \sin x_1 - kx_3) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{\Delta(x_1)}(-m^2L^2x_2^2 \cos x_1 \sin x_1 + mLkx_3 \cos x_1) \\ x_4 \\ \frac{1}{\Delta(x_1)}((I + mL^2)(mLx_2^2 \sin x_1 - kx_3) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\Delta(x_1)}(um + uM) \\ 0 \\ \frac{1}{\Delta(x_1)}(-mLu \cos x_1) \end{bmatrix}$$

$$\dot{x} = f(x) + g(u)$$

(d) Find all equilibrium points of the system.
At an equilibrium point (neglecting the input)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = f(x)$$

From the sate equation found in (c) it is clear that

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$

$$\dot{x}_3 = 0 \Rightarrow x_4 = 0$$

Plugging these values into the equations for $\dot{x}_2$ and $\dot{x}_4$ yields:

$$\dot{x}_2 = \frac{1}{\Delta(x_1)}(-m^2L^2 0^2 \cos x_1 \sin x_1 + mLkx_3 \cos x_1) = \frac{1}{\Delta(x_1)}(mLkx_3 \cos x_1)$$

$$\dot{x}_4 = \frac{1}{\Delta(x_1)}((I + mL^2)(mL0^2 \sin x_1 - kx_3) = -\frac{1}{\Delta(x_1)}(I + mL^2)kx_3$$

Due to the fact that $\frac{1}{\Delta(x_1)} \neq 0 \ \forall x_1$,

$$\dot{x}_4 = 0 \Rightarrow (I + mL^2)kx_3 = 0 \Rightarrow x_3 = 0$$

Plugging this value into the equation for $\dot{x}_2$ yields:

$$\dot{x}_2 = \frac{1}{\Delta(x_1)}(mLk0\cos x_1) = 0$$

Therefore the equilibrium points are at the points:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \end{bmatrix}, \forall \mathbf{a} \in \mathbb{R}^{\mathbf{1}}$$

Note 1 *This does not make physical sense, but after checking through the equations this is the conclusion I have reached.*

Problem 2 (Text problem 2.9) For cruise control, the longitudinal motion of a vehicle on a flat road can be modeled, with the use of Newton's second law, by the first-order differential equation

$$m\dot{v} = u - K_c sgn(v) - K_f v - K_a v^2$$

The coefficients K_c, K_f and K_a are nonnegative. $u = K_I\sigma + K_P(v_d - v)$, where v_d is the desired speed, σ is the state of the integrator $\dot{\sigma} = v_d - v$, and K_I and K_P are positive constants. I am only interested in the region $v \geq 0$.

(a) Using $x_1 = \sigma$ and $x_2 = v$ as the state variables, find the state model of the system.

$$x_1 = \sigma \Rightarrow \dot{x}_1 = \dot{\sigma} = v_d - v \tag{8}$$

$$x_2 = v \Rightarrow \dot{x}_2 = \dot{\sigma} = \frac{1}{m}(u - K_c sgn(v) - K_f v - K_a v^2) \tag{9}$$

Combining (8) and (9) and plugging in for u, v and σ yields:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} v_d - x_2 \\ \frac{1}{m}(K_I x_1 + K_P v_d - K_P x_2 - K_c sgn(x_2) - K_f x_2 - K_a x_2^2) \end{bmatrix}$$

(b) Let v_d be a positive constant. Find all equilibrium points and determine the type of each point.

Equilibrium points are defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From (8) it is clear that $\mathbf{x_2} = \mathbf{v_d}$ at the equilibrium point. From (9) I have

$$0 = \frac{1}{m}(K_I x_1 + K_P v_d - K_P v_d - K_c sgn(v_d) - K_f v_d - K_a v_d^2) \Rightarrow \mathbf{x_1} = \frac{\mathbf{K_c} + \mathbf{K_f v_d} + \mathbf{K_a v_d^2}}{\mathbf{K_I}}$$

To determine the point type I must linearize at the equilibrium point by taking the Jacobian of the state model:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -1 \\ \frac{K_I}{m} & -(K_p + K_f + 2K_a v_d) \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & - \\ + & - \end{bmatrix}$$

The eigenvalues of the Jacobian matrix will be < 0, so the point is a **stable focus**.

(c) Construct the phase portrait and discuss the qualitative behavior of the system, for the following numerical data: $m = 1500$ kg, $K_c = 110$ N, $K_f = 2.5$ n/m/sec, $K_a = 1$ N/m^2/sec^2, $K_I = 15$, $K_p = 500$, and $v_d = 30$ m/sec. **See attached paper.**

(d) Repeat part (c) when K_I is increased to 150. Compare with the behaviour in part (c). **See attached paper.**

Problem 3 (Text problem 3.1) For each of the functions $f(x)$ given next, find whether f is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz.

(1) $f(x) = x^2 + |x|$. (a) $|x|$ is not differentiable at $x = 0$, so the function **is not continuously differentiable**. (b) The function **is locally Lipschitz**. (c) The function **is continuous**. (d) The derivative is not bounded, so the function **is not globally Lipschitz**.

(2) $f(x) = x + \text{sgn}(x)$. (a, b, c, d) The function **is discontinuous so it is not continuously differentiable, locally Lipschitz or globally Lipschitz**.

(3) $f(x) = sin(x)\text{sgn}(x)$. (a) The function is not differentiable at $x = 0$, so it **is not continuously differentiable**. (b) The function is not differentiable only on a set of size 0 ($x = 0$), so it **is locally Lipschitz**. (c) The function **is continuous**. (d) The function is not differentiable only on a set of size 0 ($x = 0$) and the derivative is bounded, so it **is globally Lipschitz**.

(4) $f(x) = -x + a\sin(x)$. (a, b, c) The function **is continuously differentiable**, so it **is locally Lipschitz and continuous**. (d) The function is also bounded, so it **is globally Lipschitz**.

(5) $f(x) = -x + 2|x|$. (a) $|x|$ is not differentiable at $x = 0$, so the function **is not continuously differentiable**. (b) The function **is locally Lipschitz**. (c) The function **is continuous**. (d) The derivative is bounded so the function **is globally Lipschitz**.

(6) $f(x) = \tan(x)$. (a, b, c, d) The function **is discontinuous so it is not continuously differentiable, locally Lipschitz or globally Lipschitz**.

(7) $f(x) = \begin{bmatrix} ax_1 + \tanh(bx_1) - \tanh(bx_2) \\ ax_2 + \tanh(bx_1) + \tanh(bx_2) \end{bmatrix}$. (a, b, c) $\tanh(x)$ is continuously differentiable, so the function **is continuously differentiable, locally Lipschitz and continuous**. The derivative of $\tanh(x)$ is also bounded, so the derivative of the function is bounded and so it **is globally Lipschitz**.

(8) $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$. (a) $|x_2|$ is not differentiable at $x_2 = 0$, so the function **is not continuously differentiable**. (b, c) The function **is locally Lipschitz and continuous**. (d) The derivative of bx_1^2 is not bounded, so the function **is not globally Lipschtiz**.

Problem 4 (Text problem 3.6) Let $f(t, x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$||f(t,x)|| \leq k_1 + k_2||x||, \forall (t,x) \in [t_0,\infty) * \mathbb{R}^n$$

(a) Show that the solutions of (3.1) satisfies

$$||x(t)|| \leq ||x_0|| exp[k_2(t-t_0)] + \frac{k_1}{k_2} exp[k_2(t-t_0)] - 1 \tag{10}$$

for all $t \geq t_0$ for which the solution exists.
Plugging in for $x(t)$ in the equation for $f(t,x)$

$$||f(t,x)|| \leq k_1 + k_2(||x_0|| exp[k_2(t-t_0)] + \frac{k_1}{k_2} exp[k_2(t-t_0)] - 1)$$

$$\Rightarrow \mathbf{||f(t,x)|| \leq k_2||x_0|| exp[k_2(t-t_0)] + k_1 exp[k_2(t-t_0)]}$$

Taking the derivative of (10) yields

$$\frac{\partial ||x(t)||}{\partial t} \leq k_2||x_0|| exp[k_2(t-t_0)] + \frac{k_1}{k_2} k_2 exp[k_2(t-t_0)] = \mathbf{k_2||x_0|| exp[k_2(t-t_0)] + k_1 exp[k_2(t-t_0)]}$$

(b) Can the solution have a finite escape time?
Yes, because it is only bounded by an exponential (not globally Lipschitz) function.

Problem 5 (Text problem 4.13) For each of the following systems, show that the origin is unstable.

$$\dot{x}_1 = x_1^3 + x_1^2 x_2; \ \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3 \tag{11}$$

$$\dot{x}_1 = -x_1^3 + x_2; \ \dot{x}_2 = x_1^6 - x_2^3 \tag{12}$$

For (11) I will choose $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$, which is positive when $x_2^2 \leq x_1^2$.

$$< \nabla V(x), f(x) >= x_1 \dot{x}_1 - x_2 \dot{(x)}_2 = x_1^4 + x_1^3 x_2 + x_2^2 - x_2^3 - x_1 x_2^3 + x_2 x_1^3$$

Using Young's inequality

$$< \nabla V(x), f(x) > \geq x_1^4 + x_1^2 x_2^2 + x_1^4 + x_2^2 - x_2^3 - x_1^2 x_2^2 - x_2^4 = 2x_1^4 + x_2^2 - x_2^3 - x_2^4$$

The terms x_1^4 and x_2^2 will always be positive, and in a neighborhood of the origin the term x_2^2 will dominate the terms x_2^3 and x_2^4. Therefore, $< \nabla V(x), f(x) >> 0$, and thus the system is unstable.

For (12), I will first show that $\Gamma = 0 \leq x_1 \leq 1 \cap x_2 \geq x_1^3 \cap x_2 \leq x_1^2$ is a positively invariant set. First, I will choose $V_1(x) = x_2 - x_1^3 \geq 0$

$$< \nabla V_1(x), f(x) >= -3x_1^2 \dot{x}_1 + \dot{x}_2$$

$$= 3x_1^5 - 3x_1^2 x_2 + x_1^6 - x_2^3 \geq 3x_2 x_1 - 3x_1^2 x_2 + x_2^2 - x_2^2 = x_2^2 - x_2^2 \geq 0$$

Therefore, at the lower limit of the set the values either stay on the limit or move up into the set. Next, I will choose $V_2(x) = x_2 - x_1^2 \leq 0$

$$< \nabla V_2(x), f(x) >= -2x_1\dot{x}_1 + \dot{x}_2$$

$$= 2x_1^4 - 2x_1x_2 + x_1^6 - x_2^3 \leq 2x_1^4 - 2x_1x_2 + x_2^3 - x_2^3 = 2x_1^4 - 2x_1x_2 \leq 2x_1^2x_2 - 2x_1x_2$$

The term $2x_1x_2$ will dominate or balance the term $2x_1^2x_2$, therefore $< \nabla V_2(x), f(x) > \leq 0$ and so at the upper limit the set will either stay on the limit or move down into the set. As solutions approach the lower bound $\dot{x}_2$ approaches zero and $\dot{x}_1$ becomes negative, and as solutions approach the upper bound $\dot{x}_1$ approaches zero and $\dot{x}_2$ becomes negative. Therefore trajectories inside Γ move away from the bounds and get trapped between them.

Problem 6 (Text problem 4.15) Consider the system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3),\ \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.

At every equilibrium point

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \dot{x}_1 = 0 = x_2 \rightarrow \dot{x}_3 = 0 = x_2 - x_3 = -x_3 \Rightarrow \mathbf{x_2} = \mathbf{x_3} = \mathbf{0}$$

$$\rightarrow \dot{x}_2 = 0 = -h_1(x_1) - x_2 - h_2(x_3) = -h_1(x_1) - 0 - h_2(0) = -h_1(x_1) \Rightarrow \mathbf{x_1} = \mathbf{0}$$

Therefore, the only equilibrium point is the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)dy + \frac{x_2^2}{2} + \int_0^{x_3} h_2(y)dy$ is positive definite for all $x \in \mathbb{R}^3$.

$yh_i(y) > 0$ implies that $\int_0^{x_1} h_1(y)dy > 0,\ \int_0^{x_3} h_2(y)dy > 0$ for $x_1 \neq 0,\ x_3 \neq 0$ respectively. Also, $\frac{x_2^2}{2} > 0$ for $x_2 \neq 0$, therefore $\mathbf{V(x)} > \mathbf{0}\ \forall \mathbf{x} \in \mathbb{R}^\mathbf{3} \setminus \{\mathbf{0}\}$.

(c) Show that the origin is asymptotically stable.

$$< \nabla V(x), f(x) >= h_1(x_1)\dot{x}_1 + x_2\dot{x}_2 + h_2(x_3)\dot{x}_3$$

$$= h_1(x_1)x_2 - x_2h_1(x_1) - x_2^2 - x_2h_2(x_3) + x_2h_2(x_3) - x_3h_2(x_3) = -x_2^2 - x_3h_2(x_3) \leq 0$$

$x(t) \equiv 0$ is the only solution that can stay on S as defined in Corollary 4.1, therefore the origin is asymptotically stable.

Choosing $V_2(x) = x_2$ yields

$$< \nabla V_2(x), f(x) >= \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3)$$

At the points where $< \nabla V(x), f(x) >= 0$, $x_2 = x_3 = 0$. At these points, $< \nabla V_2(x), f(x) >= -h_1(x_1) < 0 \forall x_1 \in \mathbb{R}^1 \setminus \{0\}$, therefore, using Matrosov's theorem, the origin is asymptotically stable.

(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

If h_1 and h_2 are both radially unbounded then the origin is globally asymptotically stable.

Problem 7 (Text problem 4.18) The mass-spring system of Exercise 1.12 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

First I will find the equilibrium point(s) of the system by setting $\ddot{y} = \dot{y} = 0$

$$0 = Mg - ky \Rightarrow y = \frac{Mg}{k}$$

To shift the equilibrium point to the origin, I will choose states

$$x_1 = y - \frac{Mg}{k},\ x_2 = \dot{y} \Rightarrow \dot{x}_1 = x_2, \dot{x_2} = -\frac{1}{M}(kx_1 + c_1x_2 + c_2x_2|x_2|)$$

The system is planar, so I will choose a Lyapunov function of the form $V(x) = \alpha x_1^2 + \beta x_2^2$, which is $> 0 \forall x \in \mathbb{R}^2\backslash\{0\}$ for $\alpha, \beta > 0$. $V(x)$ is also radially unbounded.

$$< \nabla V(x), f(x) >= 2\alpha x_1\dot{x}_1 + 2\beta x_2\dot{x}_2$$

$$= 2\alpha x_1x_2 - \frac{2\beta}{M}x_2(kx_1 + c_1x_2 + c_2x_2|x_2|)$$

$$= (2\alpha - \frac{2\beta k}{M})x_1x_2 - \frac{2\beta}{M}(c_1x_2^2 + c_2x_2^2|x_2|)$$

By choosing $\alpha = 1$ and $\beta = \frac{M}{k}$, $< \nabla V(x), f(x) >= -\frac{2}{k}(c_1x_2^2 + c_2x_2^2|x_2|) < 0 \forall x \in \mathbb{R}^2$ except when $x_2 = 0$. However, $x_2 = 0$ implies that x_1 is at it's equilibrium point, therefore by Barbashin and Krasovski the equilibrium point is globally asymptotically stable.

Next I will choose $V_2(x) = x_2x_1$,

$$< \nabla V_2(x), f(x) >= x_1\dot{x}_2 + x_2\dot{x}_1 = -x_1\frac{1}{M}(kx_1 + c_1x_2 + c_2x_2|x_2|) + x_2^2$$

When $< \nabla V_2(x), f(x) >= 0$, $< \nabla V_2(x), f(x) >= -\frac{1}{M}kx_1^2 < 0$, except at the origin. Therefore, by Matrosov's theorem, the origin is globally asymptotically stable.

Problem 8 (Text problem 4.28) Consider the system

$$\dot{x}_1 = -x_1,\ \dot{x}_2 = (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2$$

(a) Show that $x = 0$ is the unique equilibrium point.

At the equilibrium point $\dot{x} = 0$

$$\Rightarrow \dot{x}_1 = 0 = -x_1 \Rightarrow \mathbf{x_1 = 0}$$

$$\Rightarrow \dot{x}_2 = 0 = (x_1x_2-1)x_2^3+(x_1x_2-1+x_1^2)x_2 = (0x_2-1)x_2^3+(0x_2-1+0^2)x_2 = -x_2^3-x_2$$

$$\Rightarrow 0 = -x_2^3 - x_2 = -x_2(x_2^2+1) \Rightarrow x_2 = 0 \; or \; x_2 = \sqrt{-1} \rightarrow \mathbf{x_2 = 0}$$

Therefore the only equilibrium point is at $x = 0$.

(b) Show, by using linearization, that $x = 0$ is asymptotically stable.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 \\ x_2^4 + x_2^2 + 2x_1x_2 & 4x_1x_2^3 - 3x_2 + 2x_1x_2 - 1 + x_1^2 \end{bmatrix}_{x_1=x_2=0}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda = -1, -1$$

Both eigenvalues of the Jacobian are negative, therefore the origin is asymptotically stable.

(c) Show that $\Gamma = \{x \in \mathbb{R}^2 | x_1x_2 \geq 2\}$ is a positively invariant set.

Choose $V(x) = x_1x_2$,

$$< \nabla V(x), f(x) >= x_2\dot{x}_1 + x_1\dot{x}_2$$

$$= -x_1x_2 + x_1(x_1x_2-1)x_2^3 + x_1(x_1x_2-1+x_1^2)x_2 \geq -2 + x_1x_2^3 + x_1x_2 + x_1^3x_2$$

$$\geq -2 + 2x_2^2 + 2 + 2x_1^2 \geq \mathbf{2x_2^2 + 2x_1^2}$$

Therefore, on the the boundary of the set $x_1x_2 \geq 2$, the trajectories move into the set ($< \nabla V(x), f(x) >> 0$ because $x_1 \neq 0,\ x_2 \neq 0$ on the set). Thus the set is positively invariant.

(d) is $x = 0$ globally asymptotically stable?

No, because trajectories starting in the set $\Gamma = \{x \in \mathbb{R}^2 | x_1x_2 \geq 2\}$ never converge to $x = 0$.