Home Work 2
ME/ECE 236 – Spring 2008

Michael Lawson

May 5, 2008

Problem 1 (Text problem 4.39) A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 - g(t)x_2$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty \quad \textit{and} \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t,x) = \frac{1}{2}(a \sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

$$V(t,x) \geq \frac{1}{2}(a \sin x_1 + x_2)^2 + [1 + a^2 - a^2](1 - \cos x_1)$$

$$V(t,x) \geq \frac{1}{2}(a \sin x_1 + x_2)^2 + (1 - \cos x_1) = W_1(x)$$

$(1 - \cos x_1) \geq 0$, therefore $W_1(x)$ is positive definite.

$$V(t,x) \leq \frac{1}{2}(a \sin x_1 + x_2)^2 + [1 + a\beta - a^2](1 - \cos x_1)$$

$$V(t,x) \leq \frac{1}{2}(a \sin x_1 + x_2)^2 + [1 + a(\beta - a)](1 - \cos x_1) = W_2(x)$$

$W_2(x)$ is positive definite because $\beta > a$. Therefore $W_1(x) \leq V(x,t) \leq W_2(x)$ and thus $V(t,x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1 - \cos x_1) + O(||x||^3)$, where $O(||x||^3)$ is a term bounded by $k||x||^3$ in some neighborhood of the origin.

$$\dot{V}(t,x) = a\dot{g}(t)(1-\cos x_1) + [(a\sin x_1 + x_2)(a\cos x_1) + (1 + ag(t) - a^2)(\sin x_1)]\dot{x}_1$$

$$+(a \sin x_1 + x_2)\dot{x}_2$$

$$= a\dot{g}(t)(1 - \cos x_1) + a^2 x_2 \sin x_1 \cos x_1 + a x_2^2 \cos x_1 + x_2 \sin x_1 + ag(t)x_2 \sin x_1$$

$$-a^2x_2\sin x_1 - a\sin^2 x_1 - x_2\sin x_1 - ag(t)x_2\sin x_1 - g(t)x_2^2$$

$$= a\dot{g}(t)(1-\cos x_1)+a^2x_2\sin x_1\cos x_1-a^2x_2\sin x_1-a\sin^2 x_1+ax_2^2\cos x_1-g(t)x_2^2$$

$$ax_2^2\cos x_1 - g(t)x_2^2 \le a|x_2|^2 - g(t)x_2^2 \le ax_2^2 - \alpha x_2^2 \le -(\alpha-a)x_2^2$$

$$\Rightarrow \dot{V}(t,x) = a\dot{g}(t)(1-\cos x_1)+a^2x_2\sin x_1\cos x_1-a^2x_2\sin x_1-a\sin^2 x_1-(\alpha-a)x_2^2$$

$$-a^2x_2\sin x_1 - a\sin^2 x_1 = -a\sin x_1(ax_2-\sin x_1) \le -2a(1-\cos x_1)$$

$$O||x^3|| := a^2x_2\sin x_1\cos x_1$$

$$\Rightarrow \mathbf{\dot{V}} \le -(\alpha - \mathbf{a})\mathbf{x_2^2} - \mathbf{a(2}-\gamma)(\mathbf{1}-\cos\mathbf{x_1}) + \mathbf{O}(||\mathbf{x}||^{\mathbf{3}})$$

(c) Show that the origin is uniformly asymptotically stable.

$$\dot{V} \le -(\alpha-a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(||x||^3) \tag{1}$$

$$a < \alpha \Rightarrow -(\alpha-a)x_2^2 < 0$$

$$\gamma < 2 \Rightarrow -a(2-\gamma)(1-\cos x_1) < 0$$

In addition, the first two terms of (1) dominate $O(||x||^3)$ near the origin; combined with (a) this shows that the origin is uniformly asymptotically stable.

Use Matrosov's theorem to show that the condition given on $\dot{g}(t)$ is not needed for uniform asymptotic stability.

I will choose $V_1(x) = \frac{1}{2}(x_1^2 + x_2^2)$, therefore

$$< \nabla V_1(x), f(x,t) >= x_1\dot{x_1} + x_2\dot{x_2} = x_1x_2 - x_2\sin x_1 - g(t)x_2^2$$

$$< \nabla V_1(x), f(x,t) >= \frac{1}{2}(\epsilon_1 x_1^2 + \frac{1}{\epsilon_1}x_2^2) - \frac{1}{2}(\epsilon_2\sin^2 x_1 + \frac{1}{\epsilon_2}x_2^2) - g(t)x_2^2$$

I will choose $\epsilon_1 = \epsilon_2 = \epsilon$, yielding

$$< \nabla V_1(x), f(x,t) >= \frac{\epsilon}{2}(x_1^2 - \sin^2 x_1) - g(t)x_2^2$$

In a neighborhood of the origin $x_1 = \sin x_1$, therefore

$$< \nabla V_1(x), f(x,t) >= -g(t)x_2^2 \le 0$$

I will choose $V_2(x) = x_1x_2$, therefore

$$< \nabla V_2(x), f(x,t) >= x_2^2 - x_1\sin x_1 - x_1g(t)x_2$$

In a neighborhood of the origin $x_1 = \sin x_1$, therefore

$$< \nabla V_2(x), f(x,t) >= x_2^2 - x_1^2 - x_1g(t)x_2$$

When $\dot{V}_1 = 0 \Rightarrow x_2 = 0 \Rightarrow \dot{V}_2 = -x_1^2 \leq 0$, therefore by Matrosov's theorem the origin is uniformly asymptotically stable.

Problem 2 (Text problem 4.42) Consider the system

$$\dot{x} = -a[I_n + S(x) + xx^T]x$$

where a is a positive constant, I_n is the n x n identity matrix, and S(x) is an x-dependent skew symmetric matrix. Show that the origin is globally exponentially stable.

I will choose $V_1(x) = x^2$, which is bounded by $\frac{1}{2}||x||^2 \leq V(x) \leq 2||x||^2$. Therefore

$$< \nabla V_1(x), f(x) >= -2x^T a[I_n + S(x) + xx^T]x = -2ax^2 - 2ax^T S(x)x - 2ax^4$$

$$= -2ax^2 - (ax^T S(x)x + ax^T S^T(x)x) - 2ax^4 = -2ax^2 - (ax^T(S(x) + S^T(x))x) - 2ax^4$$

Because $S(x)$ is a skew symmetry matrix, $S(x) + S^T(x) = 0$, therefore

$$< \nabla V_1(x), f(x) >= -2ax^2 - 2ax^4 \leq -\mathbf{2a||x||^2}$$

Therefore the origin is globally exponentially stable.

Repeat this problem with the coefficient a taken to be a nonnegative, time-varying parameter that is uniformly bounded and "persistently exciting", i.e., there exists $T > 0,\ \epsilon > 0,\ \sigma > 0$ such that

$$\epsilon \leq \int_t^{t+T} a(\tau)d\tau \leq \sigma$$

Establish uniform global exponential stability using:
1. a comparison theorem
From above I have that

$$\dot{V}_1(x) \leq -2||x||^2 = -2V$$

I will choose $\dot{u} = -2a(t)u$, therefore

$$u = u(t_0)e^{\int_{t_0}^t -2a(\tau)d\tau}$$

From above I know that $\int_{t_0}^t -2a(\tau)d\tau \leq -2\epsilon(t - t_0)$, therefore

$$u \leq u(t_0)e^{-2\epsilon(t-t_0)}$$

By the comparison theorem I have that

$$V \leq V(t_0)e^{-2\epsilon(t-t_0)}$$

Also, $V = x^2 \Rightarrow |x| = \sqrt{V}$ so

$$|x(t)| \leq \sqrt{x(t_0)}e^{-\epsilon(t-t_0)}$$

Because $|x(t)|$ is bounded by a class KL function and the conditions hold globally the system is globally exponentially stable.

2. a Lyapunov function.

From above I have that $V_1(x) = x^2,\ \dot{V}_1(x) \le -2a(t)x^2$. I will now select a second function $V_2(x,t) = -\int_t^\infty e^{t-\tau}2a(t)x^2 d\tau$, therefore

$$V_2(x,t) = x^T(-\int_t^\infty e^{t-\tau}2a(t)d\tau)x = x^T P(t)x,\ P(t) := -\int_t^\infty e^{t-\tau}2a(t)d\tau$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x}\dot{x} = V_2(x,t) + 2a(t)x^2 + 2x^T P(t)(-a(t)[I_n + S(x) + xx^T])x$$

From above I know that $-2x^T a(t)[I_n + S(x) + xx^T]x \le -2a(t)||x||^2$, therefore

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x}\dot{x} \le V_2(x,t) + 2a(t)x^2 - 2x^T P(t)a(t)x$$

Because $a(t)$ is uniformly bounded, I can choose some constant $M > 0$ such that

$$|\int_t^\infty e^{t-\tau}2a(t)d\tau| \le M\int_t^\infty e^{t-\tau} \le M$$

$$\Rightarrow \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x}\dot{x} \le x^T P(t)x + 2a(t)x^2 + 2x^T Ma(t)x$$

Also, "persistency of excitation" implies that $\int_T^{t+T} 2a(\tau)d\tau \ge 2\epsilon$, therefore

$$P(t) = -\int_t^\infty e^{t-\tau}2a(t)d\tau \le -\int_t^{t+T} e^{-T}2a(t)d\tau \le -2\epsilon e^{-T}$$

$$\Rightarrow \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x}\dot{x} \le -2\epsilon e^{-T}x^2 + 2a(t)x^2 + 2x^T Ma(t)x$$

I will no select a third function $V_3(x,t) = V_1(x) + \rho V_2(x,t),\ \rho > 0$, therefore

$$\frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial x}\dot{x} \le -2a(t)x^2 - 2\rho\epsilon e^{-T}x^2 + 2\rho a(t)x^2 + 2\rho Ma(t)x^2$$

$$\frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial x}\dot{x} \le -2a(t)x^2 - 2\rho\epsilon e^{-T}x^2 + 2\rho(1+M)a(t)x^2$$

I will choose $\rho = \frac{1}{2(1+M)}$, therefore

$$\frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial x}\dot{x} \le -a(t)x^2 - 2\frac{1}{2(1+M)}\epsilon e^{-T}x^2 \le \mathbf{-2\frac{1}{2(1+M)}}\epsilon\mathbf{e^{-T}x^2}$$

$$\Rightarrow V_3(x,t) = x^2 + \frac{1}{2(1+M)}x^T P(t)x$$

From the information given above and using similar arguments to bound $P(t)$ from before I have that $P(t) \ge -2\sigma e^{-T}$.

$$\Rightarrow \mathbf{(1 + \frac{1}{2(1+M)}(-2\sigma e^{-T})x^2 \le V_3(x,t) \le (1 + \frac{1}{2(1+M)}}(\mathbf{-2}\epsilon\mathbf{e^{-T}})\mathbf{x^2}$$

therefore $V_3(x,t)$ is a Lyapunov function that shows the exponential stability of the origin.

Problem 3 (Text problem 4.44) Consider the system

$$\dot{x}_1 = -x_1 + x_2 + (x_1^2 + x_2^2)\sin t, \quad \dot{x}_2 = -x_1 - x_2 + (x_1^2 + x_2^2)\cos t$$

Show that the origin is exponentially stable and estimate the region of attraction.

I will choose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, which satisfies $\frac{1}{2}||x||^2 \leq V(x) \leq \frac{1}{2}||x||^2$.

$$\dot{V}(x) = < \nabla V(x), f(t,x) > = x_1\dot{x}_1 + x_2\dot{x}_2$$

$$\dot{V}(x) = -x_1^2 + x_1x_2 + x_1(x_1^2 + x_2^2)\sin t - x_1x_2 - x_2^2 + x_2(x_1^2 + x_2^2)\cos t$$

$$\dot{V}(x) \leq -x_1^2 - x_2^2 + \frac{1}{2}(\epsilon_1 x_1^2 + \frac{1}{\epsilon_1}(x_1^2 + x_2^2)^2 \sin^2 t) + \frac{1}{2}(\epsilon_2 x_2^2 + \frac{1}{\epsilon_2}(x_1^2 + x_2^2)^2 \cos^2 t)$$

Choosing $\epsilon_1 = \epsilon_2 = 1$ yields

$$\dot{V}(x) \leq -\frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}(x_1^2 + x_2^2)^2$$

$$\dot{V}(x) \leq -\frac{1}{2}||x||^2 + \frac{1}{2}||x||^4 = -\frac{1}{2}(1 + ||x||^2)||x||^2$$

Choosing a $\bar{k}$ such that $\bar{k} < 1$ and $k := \frac{(1-\bar{k})}{2} \Rightarrow k > 0$ it follows that inside the ball $||x|| < 1$

$$\dot{\mathbf{V}}(\mathbf{x}) \leq -\mathbf{k}||\mathbf{x}||^{\mathbf{2}}$$

Therefore the origin is exponentially stable. My estimate of the region of attraction is the ball where the above conditions hold, which is $||\mathbf{x}|| < \mathbf{1}$. Outside of this ball the higher order positive terms dominate $\dot{V}(x)$.

Problem 4 (Text problem 4.45) Consider the system

$$\dot{x}_1 = h(t)x_2 - g(t)x_1^3, \quad \dot{x}_2 = -h(t)x_1 - g(t)x_2^3$$

where $h(t)$ and $g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0$, for all $t \geq 0$.

(a) Is the equilibrium point $x = 0$ uniformly asymptotically stable?

I will choose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, therefore

$$< \nabla V(x), f(t,x) > = x_1\dot{x_1} + x_2\dot{x_2}$$

$$= x_1h(t)x_2 - g(t)x_1^4 - x_2h(t)x_1 - g(t)x_2^4 = -g(t)(x_1^4 + x_2^4) \leq -\mathbf{k}(\mathbf{x_1^4} + \mathbf{x_2^4})$$

Therefore the origin is uniformly asymptotically stable.

(b) Is it exponentially stable?

The Lyapunov function from (a) does not meet the requirements for exponential stability. I will now check what a linearized version of the system is doing at the origin:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3g(t)x_1^2 & h(t) \\ -h(t) & -3g(t)x_2^2 \end{bmatrix} = \begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix}$$

Using the Lyapunov function from (a) on the linear system yields:

$$< \nabla V(x), f(t,x) >= x_1\dot{x_1} + x_2\dot{x_2} = -x_1 h(t) x_2 + x_2 h(t) x_1 = 0$$

Near the origin the linear system will dominate the solutions, and the above calculation shows that there are surfaces on which the linear system holds constant. I know the nonlinear systems is uniformly asymptotically stable from (a), but the above calculations on the linear version of the system at the origin imply that the nonlinear system does not converge fast enough to be exponentially stable.

(c) Is it globally uniformly asymptotically stable?

$V(x)$ from (a) is radially unbounded, therefore the system is globally asymptotically stable.

(d) Is it globally exponentially stable?

Global exponential stability is a stricter requirement than exponential stability, and in (b) I showed that the system is NOT exponentially stable. Therefore the system is not globally exponentially stable.

Problem 5 (Text problem 4.48) Consider two systems represented by $\dot{x} = f(x)$ and $\dot{x} = h(x)f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of the first system is exponentially stable if and only if the origin of the second system is exponentially stable.

Corollary 1 *(Text Corollary 4.3) Let x=0 be an equilibrium point of the nonlinear system $\dot{x} = f(x)$, where $f(x)$ is continuously differentiable in some neighborhood of $x = 0$. Let $A = \frac{\partial f}{\partial x}(0)$. Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if A is Hurwitz.*

Starting with the assumption that $f(x)$ is exponentially stable, defining $A := \frac{\partial f}{\partial x}(0)$ from corollary 1 I have that A is Hurwitz. I will define B as

$$B := h(0)\frac{\partial f}{\partial x}(0) + \frac{\partial h}{\partial x}(0)f(0) = h(0)\frac{\partial f}{\partial x}(0) = h(0)A$$

From above we know that A is Hurwitz and that $h(0)$ is a positive scalar, therefore B is also Hurwitz. Therefore, by corollary 1, if $f(x)$ is exponentially stable then $h(x)f(x)$ is exponentially stable.

To show that this is an if and only if condition I must show the reverse (that is, if $h(x)f(x)$ is exponentially stable then $f(x)$ is exponentially stable). I will start with the assumption that $h(x)f(x)$ is exponentially stable. I will define

$$A := h(0)\frac{\partial f}{\partial x}(0) + \frac{\partial h}{\partial x}(0)f(0) = h(0)\frac{\partial f}{\partial x}(0)$$

Defining $B := \frac{\partial f}{\partial x}(0)$, I have that $A = h(0)B$. A is Hurwitz (by corollary 1) and $h(0)$ is a positive scalar, therefore B must also be Hurwitz which (by corollary 1) implies that $f(x)$ is exponentially stable.

Therefore $f(x)$ is exponentially stable if and only if $h(x)f(x)$ is exponentially stable.

Problem 6 (Text problem 4.53) Consider the system $\dot{x} = f(t, x)$ and suppose there is a function $V(t, x)$ that satisfies

$$W_1(x) \leq V(t, x) \leq W_2(x), \quad \forall ||x|| \geq r > 0$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) < 0, \quad \forall ||x|| \geq r_1 \geq r$$

where $W_1(x)$ and $W_2(x)$ are continuous, positive definite functions. Show that the solutions of the system are uniformly bounded. Hint: Notice that $V(t, x)$ is not necessarily positive definite.

In the domain $D : \{x \in \mathbb{R}^n | ||x|| \geq r \; \forall r > 0\}$ $V(x)$ is decrescent and $\dot{V}(t, x) < 0$, therefore the solutions are uniformly bounded outside of an arbitrarily small neighborhood of the origin. Since D is arbitrarily close to the origin, for any $\epsilon > 0$ I can find a $\delta > 0$ such that if $||x(to)|| < \delta$ then $||x(t)|| < \epsilon$, therefore the system is uniformly stable.